3-State Potts model and automorphism of vertex operator algebra of order 3

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Abstract

In [M2], the author has defined an automorphism τ of a vertex operator algebra (VOA) of order 2 using a sub VOA isomorphic to Ising model $L(\frac{1}{2},0)$. We here define an automorphism of VOA of order 3 by using a sub VOA isomorphic to a direct sum of 3-state Potts models $L(\frac{4}{5},0)$ and an its module $L(\frac{4}{5},3)$. If V is the moonshine VOA $V^{\natural}$, the defined automorphism is a 3A element of the monster simple group.

1 Introduction

In the research of the Griess algebra $V_2^{\natural}$, Conway [C] found several idempotents called axes of the Griess algebra corresponding to elements of the monster simple group $\mathbf{M} = Aut(V^{\natural})$. It was discovered that the Griess algebra is the second primary part $(V^{\natural})_2$ of the moonshine VOA $V^{\natural}$ constructed in [FLM].

It was proved in [M2] that idempotents in the Griess algebra are halves of conformal vectors (or Virasoro elements of sub VOAs). In particular, every idempotent in the Conway's list is a half of the Virasoro element of a sub VOA isomorphic to one of the minimal discrete series of Virasoro VOA $L(n,0)$ with central charge n for some $0<n<1$. For example, a 2A-involution of the monster simple group is a half of the Virasoro element of a sub VOA isomorphic to the Ising model $L(\frac{1}{2},0)$ with central charge $\frac{1}{2}$. We note that a sub VOA W in this paper does not usually have the same Virasoro element of V. Conversely, the author showed that if a VOA V contains a sub VOA W isomorphic to the Ising model $L(\frac{1}{2},0)$, then it defines an automorphism τ_W of V of order at most 2, which is a 2A-element of the monster simple group if V is the moonshine VOA $V^{\natural}$. The definition

is very simple and done as follows:
Let e be a Virasoro element of W. As we will prove the general case in Theorem 5.1, V is a direct sum of (possibly infinite number of) irreducible W-modules since $W \cong L(\frac{1}{2},0)$ is rational. It is well known that $L(\frac{1}{2},0)$ has the exactly three irreducible modules $L(\frac{1}{2},0)$, $L(\frac{1}{2},\frac{1}{2})$ and $L(\frac{1}{2},\frac{1}{16})$. We can define an automorphism τ_e by

$$\tau_e : \begin{cases} \ \ 1 & on \quad W\text{-submodules isomorphic to } L(\frac{1}{2},0) \text{ or } L(\frac{1}{2},\frac{1}{2}) \\ -1 & on \quad W\text{-submodules isomorphic to } L(\frac{1}{2},\frac{1}{16}) \end{cases} .$$

Throughout this paper, we will use the similar notation in order to define an endomorphism of V by a sub VOA W and we will omit "W-submodules isomorphic to" from the definition of automorphisms in order to simplify the notation. In the Conway's list, an idempotent for a 3A element is a half of the Virasoro element of a sub VOA isomorphic to $L(\frac{4}{5},0)$ with central charge $\frac{4}{5}$. So it is natural for us to expect an automorphism g (of order 3) defined by a sub VOA isomorphic to $L(\frac{4}{5},0)$, where $L(\frac{4}{5},0)$ is the third of the discrete series of minimal Virasoro vertex operator algebras called 3-state Potts model. It is a rational VOA and has the exactly ten irreducible modules :

$$\begin{array}{l} L(\frac{4}{5},0), L(\frac{4}{5},\frac{1}{8}), L(\frac{4}{5},\frac{2}{3}), L(\frac{4}{5},\frac{13}{8}), L(\frac{4}{5},3), \\ L(\frac{4}{5},\frac{2}{5}), L(\frac{4}{5},\frac{1}{40}), L(\frac{4}{5},\frac{1}{15}), L(\frac{4}{5},\frac{21}{40}), L(\frac{4}{5},\frac{7}{5}). \end{array}$$

As we showed in [M2], if a VOA V contains a sub VOA $W \cong L(\frac{4}{5},0)$, then we can define an automorphism σ_W of at most 2 given by

$$\sigma_W : \begin{cases} \ \ 1 & on \quad L(\frac{4}{5},0), L(\frac{4}{5},3), L(\frac{4}{5},\frac{13}{8}), L(\frac{4}{5},\frac{2}{5}), L(\frac{4}{5},\frac{1}{15}), L(\frac{4}{5},\frac{7}{5}) \\ -1 & on \quad L(\frac{4}{5},\frac{1}{8}), L(\frac{4}{5},\frac{13}{8}), L(\frac{4}{5},\frac{1}{40)}), L(\frac{4}{5},\frac{21}{40}) \end{cases} .$$

We note that we can't observe this automorphism in the moonshine VOA. Namely, $V^\natural$ does not contain any submodules of the second lines. Under such a situation, we want to define a 3A automorphism τ_W of V by a sub VOA W isomorphic to $L(\frac{4}{5},0)$. It is clear that it is not enough to think of only $L(\frac{4}{5},0)$ because there is no difference between the eigenspaces of τ with eigenvalues $e^{2\pi i/3}$ and $e^{4\pi i/3}$. Namely, let V^1 and V^2 be eigenspaces of τ with eigenvalue $e^{2\pi i/3}$ and $e^{4\pi i/3}$, then they are isomorphic as $L(\frac{4}{5},0)$-modules. However, Dong and Mason [DM] showed a wonderful result that V^1 and V^2 are not isomorphic as $V^{<\tau_W>}$-modules, where $V^{<\tau_W>}$ is the space of τ_W-invariants. So we have to think of a bigger sub VOA. What is the difference between $V^{<\tau_W>}$ and $L(\frac{4}{5},0)$? Recently, Kitazume, Yamada and the author have constructed a new class of VOAs by using codes over $\mathbb{Z}_3$ in [KMY]. The interesting point is that they used a VOA isomorphic to $L(\frac{4}{5},0) \oplus L(\frac{4}{5},3)$ as a sub VOA corresponding to $0 \in \mathbb{Z}_3$.

This is our key point and the main result in this paper is to show that if V contains a sub VOA W isomorphic to $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$, we can define a triality automorphism τ_W of V (of order 3 or possibly 1).

In order to define the automorphism, we need quote several results from [KMY]. They classified the irreducible modules of $W(0)=L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$. Namely,

Theorem 1.1 ([KMY]) *$L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$ is a rational VOA and it has the exactly six irreducible modules:*

$$W(0), W(\frac{2}{5}), W(\frac{2}{3},+), W(\frac{1}{15},+), W(\frac{2}{3},-), W(\frac{1}{15},-).$$

Here h in $W(h)$ or $W(h,\pm)$ denotes the lowest degree and $W(k,-)$ is a contragredient (dual) module of $W(k,+)$ for $k=\frac{2}{3},\frac{1}{15}$. In particular,

$$\begin{aligned}
&W(0)\cong L(\tfrac{4}{5},0)\oplus L(\tfrac{4}{5},3),\\
&W(\tfrac{2}{5})\cong L(\tfrac{4}{5},\tfrac{2}{5})\oplus L(\tfrac{4}{5},\tfrac{7}{5}),\\
&W(\tfrac{2}{3},+)\cong L(\tfrac{4}{5},\tfrac{2}{3}),\\
&W(\tfrac{2}{3},-)\cong L(\tfrac{4}{5},\tfrac{2}{3}),\\
&W(\tfrac{1}{15},+)\cong L(\tfrac{4}{5},\tfrac{1}{15}),\\
&W(\tfrac{1}{15},-)\cong L(\tfrac{4}{5},\tfrac{1}{15}),
\end{aligned}$$

as $L(\frac{4}{5},0)$-modules.

Using the notation in the above theorem, we have:

Theorem A *If a VOA V contains a sub VOA W isomorphic to $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$, then an endomorphism τ_W of V defined by*

$$\tau_W:\begin{cases} 1 & \text{on} \quad W(0) \text{ and } W(\frac{2}{5}) \\ e^{2\pi i/3} & \text{on} \quad W(\frac{2}{3},+) \text{ and } W(\frac{1}{15},+) \\ e^{4\pi i/3} & \text{on} \quad W(\frac{2}{3},-) \text{ and } W(\frac{1}{15},-) \end{cases}$$

is an automorphism of V.

In order to tell the difference between $W(h,+)$ and $W(h,-)$ for $h=\frac{2}{3},\frac{1}{15}$, we have to explain the actions of the lowest degree vector of $L(\frac{4}{5},3)$ since both $W(h,\pm)$ are isomorphic to $L(\frac{4}{5},h)$ as $L(\frac{4}{5},0)$-modules. However, we will take an easier way to avoid such a complicated job. We only note that if we fix $W(\frac{2}{3},+)$, then $W(\frac{1}{15},\pm)$ and $W(\frac{2}{3},-)$ are uniquely determined by the fusion rule

$$W(\frac{2}{3},\pm)\times W(\frac{2}{5})=W(\frac{1}{15},\pm).$$

We don't distinguish between $W(\frac{2}{3},+)$ and $W(\frac{2}{3},-)$, but if we switch them then we shall define τ_W^{-1}.

Let T be a sub VOA of W isomorphic to $L(\frac{4}{5},0)$. If $\tau=1$, then all T-submodules of V are isomorphic to $L(\frac{4}{5},0)$, $L(\frac{4}{5},3)$, $L(\frac{4}{5},\frac{2}{5})$ or $L(\frac{4}{5},\frac{7}{5})$. In this case, we can define another automorphism μ_T of V as follows:

Theorem B *Assume that V contains a sub VOA T isomorphic to $L(\frac{4}{5},0)$ and all T-submodules of V are isomorphic to $L(\frac{4}{5},0)$, $L(\frac{4}{5},3)$, $L(\frac{4}{5},\frac{2}{5})$ or $L(\frac{4}{5},\frac{7}{5})$. Then the endomorphism μ_T defined by*

$$\mu_T : \begin{cases} 1 & \text{on} \quad L(\frac{4}{5},0) \text{ and } L(\frac{4}{5},\frac{7}{5}) \\ -1 & \text{on} \quad L(\frac{4}{5},3) \text{ and } L(\frac{4}{5},\frac{2}{5}) \end{cases}$$

is an automorphism of V.

The proofs of these theorems are based on Theorem 2.1 (Proposition 4.4 in [M1]). Namely, it is sufficient to show that the fusion rules among the irreducible $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$-modules commutes with τ_W. For an example, we know the fusion rules among irreducible $L(\frac{4}{5},0)$-modules (Table A), which proves Theorem B. Therefore, the main thing we will do in this paper is to determine the fusion rules among the irreducible $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$-modules.

In this paper, we often view V as a W-module (or an infinite direct sum of W-modules) if V contains a sub VOA W. This is not obvious since one of the axioms of VOA-modules expects the grade keeping operator e_1 of Virasoro element e of W to act on V diagonally. We will prove in §4 that this is generally true for a rational sub VOA W.

2 Preliminary results and a generalized VOA constructed from a lattice

Throughout this paper, $W(0)$ denotes a VOA isomorphic to $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$. Since we will treat only a rational VOA V isomorphic to $L(\frac{4}{5},0)$ or $W(0)$, the tensor products of two V-modules M^1 and M^2 are always well-defined and it is isomorphic to $\oplus_U N^U_{M^1,M^2}U$, where U runs over the all irreducible V-modules. Therefore, it is equal to the fusion rule in our case and so we will use the same notation $M^1\times M^2$ to denote the tensor product.

Since $L(\frac{4}{5},0)\subseteq W(0)$, all $W(0)$-modules are $L(\frac{4}{5},0)$-modules. Using this fact, we will give an upperbound of the fusion rules of $W(0)$-modules. Using exactly the same proof, we can modify Proposition 11.9 in [DL] into the following theorem.

Theorem 2.1 ([DL]) *Let W^1, W^2, W^3 be V-modules and assume that W^1, W^2 have no proper submodules containing v^1 and v^2, respectively. Let $I \in I\begin{pmatrix} & W^3 & \\ W^1 & & W^2\end{pmatrix}$. If $I(v^1,z)v^2 = 0$, then $I(\cdot, z) = 0$.*

In the case where $W^1 = W(h,+) \oplus W(h,-)$ for $h = \frac{1}{15}, \frac{2}{3}$, W^1 has no proper submodule containing $U^1 = \{(v, \phi(v)) \in W(h,+) \oplus W(h,-)\}$, where $\phi : W(h,+) \to W(h,-)$ is a $L(\frac{4}{5}, 0)$-isomorphism. Therefore, we have the following theorem:

Lemma 2.1 *The maps*

$$\begin{aligned}
\phi^1 &: I_{W(0)}\begin{pmatrix} & W^3 & \\ W(i) & & W(j)\end{pmatrix} \to I_{L(\frac{4}{5},0)}\begin{pmatrix} & W^3 & \\ L(\frac{4}{5},i) & & L(\frac{4}{5},j)\end{pmatrix}, \\
\phi^2 &: I_{W(0)}\begin{pmatrix} & W^3 & \\ W(h,+)\oplus W(h,-) & & W(k,+)\oplus W(k,-)\end{pmatrix} \to I_{L(\frac{4}{5},0)}\begin{pmatrix} & W^3 & \\ L(\frac{4}{5},h) & & L(\frac{4}{5},k)\end{pmatrix}
\end{aligned}$$

and

$$\phi^3 : I_{W(0)}\begin{pmatrix} & W^3 & \\ W(h,+)\oplus W(h,-) & & W(i)\end{pmatrix} \to I_{L(\frac{4}{5},0)}\begin{pmatrix} & W^3 & \\ L(\frac{4}{5},h) & & L(\frac{4}{5},j)\end{pmatrix}$$

induced by the restrictions are all injective for $i, j = 0, \frac{2}{5}$ and $h, k = \frac{2}{3}, \frac{1}{15}$.

Throughout this paper, $N^{W^3}_{W^1,W^2}$ denotes $\dim I\begin{pmatrix} & W^3 & \\ W^1 & & W^2\end{pmatrix}$.

It is known that

$$N^{W^3}_{W^1,W^2} = N^{W^3}_{W^2,W^1} = N^{(W^1)'}_{W^2,(W^3)'}$$

where W' denotes the contragredient (dual) module of W, (see Proposition 5.5.2 in [FHL]). We note that $N^{W^1}_{W(0),W^1} = 1$ and $N^{W^2}_{W(0),W^1} = 0$ for $W^1 \ncong W^2$. Let $k' = 3, \frac{7}{5}$ for $k = 0, \frac{2}{5}$, respectively. Namely, $W(k) \cong L(\frac{4}{5},k) \oplus L(\frac{4}{5},k')$ as $L(\frac{4}{5},0)$-modules. By the above lemma and the fusion rules of the irreducible $L(\frac{4}{5},0)$-modules (see Table A), we have the following lemma.

Lemma 2.2

$$\begin{aligned}
&N^{W(k)}_{W(i),W(j)} \le N^{L(\frac{4}{5},k)\oplus L(\frac{4}{5},k')}_{L(\frac{4}{5},i),L(\frac{4}{5},j)} \le 1 \\
&N^{W(k,\pm)}_{W(i),W(j)} \le N^{L(\frac{4}{5},k)}_{L(\frac{4}{5},i),L(\frac{4}{5},j)} = 0 \\
&N^{W(k)}_{W(i,+)\oplus W(i,-),W(j,+)\oplus W(j,-)} \le N^{L(\frac{4}{5},k)\oplus L(\frac{4}{5},k')}_{L(\frac{4}{5},i),L(\frac{4}{5},j)} \le 2 \\
&N^{W(k,\pm)}_{W(i,+)\oplus W(i,-),W(j,+)\oplus W(j,-)} \le N^{L(\frac{4}{5},k)}_{L(\frac{4}{5},i),L(\frac{4}{5},j)} \le 1
\end{aligned}$$

Let's explain how we determine the fusion rules of $W(0)$-modules. Since all VOAs in this paper are rational, we identify the fusion product and the tensor product. Namely,

we will see $W^1 \times W^2$ as a $W(0)$-module for $W(0)$-modules W^1 and W^2. Let T be a sub VOA of $W(0)$ isomorphic to $L(\frac{4}{5},0)$. For a $W(0)$-module W, it is also a T-module and we denote them by $(W)_T$. As $W(0)$-modules, we have a fusion product $W^1 \times W^2$. Also viewing W^1 and W^2 as $L(\frac{4}{5},0)$-modules, we have a fusion product $(W^1)_T \times (W^2)_T$. The above two lemmas tell that there is an injective $W(0)$-homomorphism π of $(W^1 \times W^2)$ into $(W^1)_T \times (W^2)_T$. We note that all fusion rules $N^{L(\frac{4}{5},i)}_{L(\frac{4}{5},j),L(\frac{4}{5},h)}$ are less than or equal to 1. We will next show $N^{W1}_{W^2,W^3} \neq 0$ for desired $W(0)$-modules W^1, W^2, W^3 so that the above injection π of $(W^1 \times W^2)$ into $(W^1)_T \times (W^2)_T$ is an isomorphism. Therefore, we can determine the fusion rules.

In order to show $N^{W1}_{W^2,W^3} \neq 0$ for some W^1, W^2, W^3, we will use a (generalized) VOA V_L constructed from a lattice $L = \frac{\sqrt{2}}{3}A_2$. First, we quote a construction of V_S for $S \subseteq \mathbb{R}L$ for [FLM]. We note that we don't need a group extension here since all the values of inner products of elements of L are in $2\mathbb{Z}[1/3]$.

Let L be a lattice. Viewing $H = \mathbb{C}L$ as a commutative Lie algebra with a bilinear form $<,>$, we define the affine Lie algebra

$$\hat{H} = H[t,t^{-1}] + \mathbb{C}k$$

associated with H and the symmetric tensor algebra $M(1) = S(\hat{H}^-)$ of $\hat{H}^-$, where $\hat{H}^- = H[t^{-1}]t$. As in [FLM], we shall define the Fock space $V_L = \oplus_{a\in L} M(1)\mathbf{e^a}$ with the vacuum $\mathbf{1} = \mathbf{e^0}$ and the vertex operators $Y(*,z)$ as follows:

The vertex operator of $\mathbf{e}^a$ is given by

$$Y(\mathbf{e^a},\mathbf{z}) = \mathbf{exp}\left(\sum_{\mathbf{n}\in\mathbb{Z}_+} \frac{\mathbf{a(-n)}}{\mathbf{n}}\mathbf{z^n}\right) \mathbf{exp}\left(\sum_{\mathbf{n}\in\mathbb{Z}_+} \frac{\mathbf{a(n)}}{\mathbf{-n}}\mathbf{z^{-n}}\right) \mathbf{e^a z^a}.$$

and that of $a(-1)\mathbf{e^0}$ is

$$Y(a(-1)\mathbf{e^0},\mathbf{z}) = \mathbf{a(z)} = \sum \mathbf{a(n)z^{-n-1}}.$$

The vertex operators of other elements are defined by the n-th normal product:

$$Y(a(n)v,z) = a(z)_n Y(v,z) = Res_x\{(x-z)^n a(x)Y(v,z) - (-z+x)^n Y(v,z)a(x)\}.$$

Here the operator of $a \otimes t^n$ on $M(1)\mathbf{e^b}$ are denoted by $a(n)$ and

$$\begin{aligned}
&a(n)\mathbf{e^b} = \mathbf{0} \text{ for } \mathbf{n} > \mathbf{0},\\
&a(0)\mathbf{e^b} = <\mathbf{a},\mathbf{b}> \mathbf{e^b},\\
&e^a\mathbf{e^b} = \mathbf{e^{a+b}},\\
&z^a\mathbf{e^b} = \mathbf{e^b z^{<a,b>}},\\
&(x+y)^n = \textstyle\sum_{i=0}^{\infty} \binom{n}{i} x^{n-i}y^i \text{ and}\\
&\binom{n}{i} = \tfrac{n(n-1)\cdots(n-i+1)}{i!}.
\end{aligned}$$

We note that the above definition of vertex operator is very general, that is, it is well defined for any $v \in V_{\mathbb{R}\otimes L}$ and so we may think

$$Y(v,z) \in End(V_{\mathbb{R}\otimes L})((z,z^{-1}))$$

for $v \in V_{\mathbb{R}\otimes_{\mathbb{Z}} L}$, where $P((z,z^{-1}))$ denotes $\{\sum_{n\in\mathbb{C}} a_n z^n : a_n \in P\}$ for any set P. In particular,

$$Y(v,z)u \in V_{\mathbb{R}\otimes L}[[z,z^{-1}]]z^{\langle a,b\rangle}$$

for $v \in M(1)e^a$ and $u \in M(1)e^b$ and $a,b \in \mathbb{R}L$. Set $\mathbf{1} = \mathbf{e^0}$. It is worth to note that if we set $Y(v,z) = \sum_{n\in\mathbb{R}} v_n z^{-n-1}$, then $v_{-1}\mathbf{e^0} = \mathbf{v}$ for any $v \in \mathbb{R}\otimes L$. Also, for the Virasoro element, we set

$$w = \sum v^i(-1)^1 e^0$$

where $\{v^1,...,v^k\}$ is an orthonormal basis of $\mathbb{R}L$.

For any subset S of $\mathbb{R}L$, we can define

$$V_S = \oplus_{a\in S} M(1)\mathbf{e^a}.$$

The followings are obtained in Chapter 4 of [FLM].

Lemma 2.3 ([FLM])

$$\begin{array}{lll}
[\sum a(n)x^{-n-1}, Y(\mathbf{e^b},\mathbf{z})] \sim \mathbf{0} & \textit{for any } a,b \in L, & \\
Y(\mathbf{e^a},\mathbf{x})\mathbf{Y}(\mathbf{e^b},\mathbf{z}) \sim \mathbf{Y}(\mathbf{e^b},\mathbf{z})\mathbf{Y}(\mathbf{e^a},\mathbf{x}) & \textit{for } <a,b>\equiv 2 & \pmod 2, \\
Y(\mathbf{e^a},\mathbf{x})\mathbf{Y}(\mathbf{e^b},\mathbf{z}) \sim -\mathbf{Y}(\mathbf{e^b},\mathbf{z})\mathbf{Y}(\mathbf{e^a},\mathbf{z}) & \textit{for } <a,b>\equiv 1 & \pmod 2.
\end{array}$$

where $a(x,z) \sim b(x,z)$ means $(z-x)^m(a(x,z)-b(x,z)) = 0$ for some $m \in \mathbb{Z}$. Especially, if $\langle a,b\rangle \in 2\mathbb{Z}$, then $[Y(v,z),Y(u,x)] \sim 0$ for $v \in M(1)e^a, u \in M(1)e^b$.

In [KMY], they studied the structure of the VOA $M^0 = V_L$ and its modules for the lattice $L = \sqrt{2}A_2$. Namely, let $\langle x,y\rangle = -2$, $\langle x,x\rangle = \langle y,y\rangle = 4$ and set $L = \mathbb{Z}x + \mathbb{Z}y$ be a lattice (of type $\sqrt{2}A_2$). It is easy to see that $M^1 = V_{\frac{x+2y}{3}+L}$ and $M^2 = V_{\frac{2x+y}{3}+L}$ are V_L-modules. Set

$$M = M^0 \oplus M^1 \oplus M^2.$$

We note that M is closed under the operators u_n of $u \in M$. It is proved by [DLMN] that the Virasoro element w of V_L is an orthogonal sum of three conformal vectors w^1, w^2, and w^3 with central charges $\frac{1}{2}$, $\frac{7}{10}$, and $\frac{4}{5}$, respectively. Namely, V_L contains a sub VOA T isomorphic to $L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\otimes L(\frac{4}{5},0)$. Viewing V as a T-module, it is a direct sum of irreducible modules of T and each irreducible T-module is isomorphic to

$L(\frac{1}{2},h_1)\otimes L(\frac{7}{10},h_2)\otimes L(\frac{4}{5},h_3)$ for some h_1,h_2,h_3.

In the following argument, we recall the study in [KMY].

It is clear that $(V_L)_1=\mathbb{C}x(-1)e^0+\mathbb{C}y(-1)e^0$. The sum of all subspaces of $M^0=V_L$ isomorphic to $L(\frac{1}{2},0)\otimes L(\frac{7}{10},k^1)\otimes L(\frac{4}{5},k^2)$ for any k^1,k^2 is isomorphic to a direct sum of

$$T^1=L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\otimes\Big(L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)\Big)$$

and

$$T^2=L(\frac{1}{2},0)\otimes L(\frac{7}{10},\frac{3}{5})\otimes\Big(L(\frac{4}{5},\frac{2}{5})\oplus L(\frac{4}{5},\frac{7}{5})\Big).$$

Also, the sum of subspaces of M^1 isomorphic to $L(\frac{1}{2},0)\otimes L(\frac{7}{10},k^1)\otimes L(\frac{4}{5},k^2)$ is a direct sum of

$$\begin{array}{l}T^{3+}=L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\otimes L(\frac{4}{5},\frac{2}{3})\\ T^{4+}=L(\frac{1}{2},0)\otimes L(\frac{7}{10},\frac{3}{5})\otimes L(\frac{4}{5},\frac{1}{15}).\end{array}$$

Similarly, the sum of subspaces of M^2 isomorphic to $L(\frac{1}{2},0)\otimes L(\frac{7}{10},k^1)\otimes L(\frac{4}{5},k^2)$ is a direct sum of

$$\begin{array}{l}T^{3-}=L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\otimes L(\frac{4}{5},\frac{2}{3})\\ T^{4-}=L(\frac{1}{2},0)\otimes L(\frac{7}{10},\frac{3}{5})\otimes L(\frac{4}{5},\frac{1}{15}).\end{array}$$

It is easy to see that $T^{n\pm}$ are contragredient (dual) modules of $T^{n\mp}$ by the natural inner product of $V_{\mathbb{Z}(\frac{x+2y}{3})+L}$.

Since $\langle L,\frac{x+2y}{3}+L\rangle\subseteq 2\mathbb{Z}$, for $v\in M^g$ with $g\in\mathbb{Z}_3$, the vertex operator $Y(v,z)$ satisfies $L(-1)$-derivative property and the local commutativity with all vertex operators $Y(u,z)$ of $u\in V_L$. By applying it to M^h for $h\in\mathbb{Z}_3$, we have an intertwining operator $Y(*,z)\in I\begin{pmatrix} & M^{h+g} & \\ M^g & & M^h\end{pmatrix}$.

When we view these intertwining operators as intertwining operators among $L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\otimes\{L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)\}$-submodules and then as intertwining operators among $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$-modules, the following theorem is very useful.

Theorem 2.2 ([M1]) *Let W be a sub VOA of V which may have a different Virasoro element $e\in V_2$. Assume that W is rational. Let M^1 and M^2 be irreducible W-submodules of V. Set*

$$M(M^1,M^2)=\sum_{v\in M^1,u\in M^2,m\in\mathbb{Z}}\mathbb{C}v(m)u.$$

Then $M(M^1,M^2)$ is a W-module and we have $I_W\binom{M^3}{M^1\ \ M^2}\neq 0$ for any irreducible W-submodule M^3 of $M(M^1,M^2)$. In particular, let $\{W^i : i\in I\}$ be the set of distinct

irreducible W-modules and $V = \oplus V^i$ be the decomposition into the direct sum of homogeneous W-modules V^i, where V^i is the sum of all irreducible W-submodules of V isomorphic to W^i. For $u \in V$, let $u = \sum u^i$, where $u^i \in V^i$. If there are $v \in V^i$, $u \in V^j$, $n \in \mathbb{Z}$ such that $(v_n u)^k \neq 0$, then $I\begin{pmatrix} & V^k & \\ V^i & & V^j \end{pmatrix} \neq 0$.

We should note that by Theorem 2.2 and the fusion rules $L(\frac{1}{2},0) \times L(\frac{1}{2},0) = L(\frac{1}{2},0)$, $T^1 \oplus T^2 \oplus T^{3+} \oplus T^{3-} \oplus T^{4+} \oplus T^{4-}$ is closed by the products.

Since $W(h,-)$ and $W(i)$ are contragredient (dual) modules of $W(h,+)$ and $W(i)$, respectively, we have:

$$N^{W(0)}_{W(h,\pm),W(h,\pm)} \neq 0 \quad \text{and} \quad N^{W(0)}_{W(i),W(i)} \neq 0. \tag{2.1}$$

for $h = 0, \frac{2}{5}$ and $i = \frac{2}{3}, \frac{1}{15}$.

It is easy to check that

$$\begin{aligned}
(T^2)_1 &= \mathbb{C}(x+2y)(-1)e^0 \\
(T^1)_2 &= \mathbb{C}(3x(-1)^2e^0 + (x+2y)(-1)^2e^0) \\
(T^{3+})_{2/3} &= \mathbb{C}(e^{(x+2y)/3} + e^{(x-y)/3} + e^{(-2x-y)/3}) \\
(T^{4+})_{2/3} &= \mathbb{C}(2e^{(x+2y)/3} - e^{(x-y)/3} - e^{(-2x-y)/3})
\end{aligned}$$

Since

$$((x+2y)(-1)e^0)_{-1}(x+2y)(-1)e^0 = (x+2y)(-1)^2e^0 \notin (T^1)_2$$

we have

$$N^{W(\frac{2}{5})}_{W(\frac{2}{5}),W(\frac{2}{5})} \neq 0. \tag{2.2}$$

Also, since

$$(x+2y)(-1)\{\lambda e^{(x+2y)/3} + \mu(e^{(x-y)/3} + e^{(-2x-y)/3})\} = 4\lambda e^{(x+2y)/3} - 2\mu(e^{(x+2y)/3} + e^{(-2x-y)/3}),$$

we have :

$$N^{W(\frac{1}{15},+)}_{W(\frac{2}{5}),W(\frac{2}{3},+)} = N^{W(\frac{2}{5})}_{W(\frac{2}{3},+),W(\frac{1}{15},-)} = N^{W(\frac{2}{3},-)}_{W(\frac{2}{5}),W(\frac{1}{15},-)} \neq 0 \tag{2.3}$$

and

$$N^{W(\frac{1}{15},+)}_{W(\frac{2}{5}),W(\frac{1}{15},+)} = N^{W(\frac{2}{5})}_{W(\frac{1}{15},+),W(\frac{1}{15},-)} \neq 0. \tag{2.4}$$

Similarly,

$$N^{W(\frac{1}{15},-)}_{W(\frac{2}{5}),W(\frac{2}{3},-)} = N^{W(\frac{2}{5})}_{W(\frac{2}{3},-),W(\frac{1}{15},+)} = N^{W(\frac{2}{3},+)}_{W(\frac{2}{5}),W(\frac{1}{15},+)} \neq 0 \tag{2.5}$$

and

$$N^{W(\frac{1}{15},-)}_{W(\frac{2}{5}),W(\frac{1}{15},-)} = N^{W(\frac{2}{5})}_{W(\frac{1}{15},-),W(\frac{1}{15},+)} \neq 0. \tag{2.6}$$

It follows from the direct calculations that $e^{(x+2y)/3}+e^{(x-y)/3}+e^{(-2x-y)/3}$ and $2e^{(x+2y)/3}-(e^{(x-y)/3}+e^{(-2x-y)/3})$ are lowest degree vectors of $L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\otimes L(\frac{4}{5},\frac{2}{3})\subseteq M^1$ and $L(\frac{1}{2},0)\otimes L(\frac{7}{10},\frac{3}{5})\otimes L(\frac{4}{5},\frac{1}{15})\subseteq M^1$, respectively. Similarly, $e^{(-x-2y)/3}+e^{(-x+y)/3}+e^{(2x+y)/3}$ and $2e^{(-x-2y)/3}-(e^{(-x+y)/3}+e^{(2x+y)/3})$ are lowest degree vectors of $L(\frac{1}{2},0)\otimes L(\frac{7}{10},0)\otimes L(\frac{4}{5},\frac{2}{3})\subseteq M^2$ and $L(\frac{1}{2},0)\otimes L(\frac{7}{10},\frac{3}{5})\otimes L(\frac{4}{5},\frac{1}{15})\subseteq M^2$, respectively. Also, for $u=\alpha e^{(x+2y)/3}+\beta(e^{(x-y)/3}+e^{(-2x-y)/3})$ and $v=\lambda e^{(x+2y)/3}+\mu(e^{(x-y)/3}+e^{(-2x-y)/3})$, we have $u_{-1/3}v=2\beta\mu e^{(x+2y)/3}+\beta\lambda+\alpha\mu(e^{(x-y)/3}+e^{(-2x-y)/3})$, where $u_{-1/3}$ is the grade keeping operator of u. Hence, we have

$$N_{W(\frac{2}{3},\pm),W(\frac{2}{3},\pm)}^{W(\frac{2}{3},\mp)}\neq 0, \tag{2.7}$$

$$N_{W(\frac{1}{15},\pm),W(\frac{1}{15},\pm)}^{W(\frac{2}{3},\mp)}=N_{W(\frac{1}{15},\pm),W(\frac{2}{3},\pm)}^{W(\frac{1}{15},\mp)}\neq 0, \tag{2.8}$$

$$N_{W(\frac{1}{15},\pm),W(\frac{1}{15},\pm)}^{W(\frac{1}{15},\mp)}\neq 0. \tag{2.9}$$

3 Fusion rule

We first list the fusion rules among $L(\frac{4}{5},0)$-modules $L(\frac{4}{5},0)$, $L(\frac{4}{5},3)$, $L(\frac{4}{5},2/5)$, $L(\frac{4}{5},7/5)$, $L(\frac{4}{5},\frac{2}{3})$ and $L(\frac{4}{5},\frac{1}{15})$, which are the only irreducible modules we need in this section. For the fusion rules for the remaining cases, see Appendix.

Table A

0	$\frac{2}{5}$	$\frac{7}{5}$	$\frac{1}{15}$	3	$\frac{2}{3}$
$\frac{2}{5}$	$0:\frac{7}{5}$	$\frac{2}{5}:3$	$\frac{1}{15}:\frac{2}{3}$	$\frac{7}{5}$	$\frac{1}{15}$
$\frac{7}{5}$	$\frac{2}{5}:3$	$0:\frac{7}{5}$	$\frac{2}{3}:\frac{1}{15}$	$\frac{2}{5}$	$\frac{1}{15}$
$\frac{1}{15}$	$\frac{1}{15}:\frac{2}{3}$	$\frac{2}{3}:\frac{1}{15}$	$0:\frac{7}{5}:\frac{2}{3}:\frac{1}{15}:3:\frac{2}{5}$	$\frac{1}{15}$	$\frac{2}{5}:\frac{1}{15}:\frac{7}{5}$
3	$\frac{7}{5}$	$\frac{2}{5}$	$\frac{1}{15}$	0	$\frac{2}{3}$
$\frac{2}{3}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{5}:\frac{1}{15}:\frac{7}{5}$	$\frac{2}{3}$	$0:\frac{2}{3}:3$

In the table, the number h denotes $L(\frac{4}{5},h)$ and $h:\cdots:k$ denotes $L(\frac{4}{5},h)+\cdots+L(\frac{4}{5},k)$.

By (2.5) and (2.6), $N_{W(h,\pm),W(h,\pm)}^{W(h,\mp)}\neq 0$. Hence, by the fusion rules $L(\frac{4}{5},\frac{2}{3})\times L(\frac{4}{5},\frac{2}{3})=L(\frac{4}{5},0)+L(\frac{4}{5},3)+L(\frac{4}{5},\frac{2}{3})$ of $L(\frac{4}{5},0)$-modules and (2.1) and (2.7), we have

$$\begin{aligned} W(\tfrac{2}{3},\pm)\times W(\tfrac{2}{3},\pm)&=W(\tfrac{2}{3},\mp)\\ W(\tfrac{2}{3},\pm)\times W(\tfrac{2}{3},\mp)&=W(0). \end{aligned} \tag{3.1}$$

Similarly, by the fusion rules $L(\frac{4}{5},\frac{1}{15})\times L(\frac{4}{5},\frac{1}{15})=L(\frac{4}{5},0)+L(\frac{4}{5},3)+L(\frac{4}{5},\frac{2}{3})+L(\frac{4}{5},\frac{1}{15})+L(\frac{4}{5},\frac{2}{5})+L(\frac{4}{5},\frac{7}{5})$ of $L(\frac{4}{5},0)$-modules and (2.1),(2.4),(2.8) and (2.9), we have

$$\begin{aligned} W(\tfrac{1}{15},\pm)\times W(\tfrac{1}{15},\pm)&=W(\tfrac{1}{15},\mp)+W(\tfrac{2}{3},\mp)\\ W(\tfrac{1}{15},\pm)\times W(\tfrac{1}{15},\mp)&=W(0)+W(\tfrac{2}{5}) \end{aligned} \tag{3.2}$$

By the fusion rules $L(\frac{4}{5},\frac{2}{5})\times L(\frac{4}{5},\frac{2}{3})=L(\frac{4}{5},\frac{1}{15})$ and (2.3) and (2.5), we have

$$W(\frac{2}{5})\times W(\frac{2}{3},\pm)=W(\frac{1}{15},\pm). \tag{3.3}$$

By the fusion rules $L(\frac{4}{5},\frac{2}{5})\times L(\frac{4}{5},\frac{1}{15})=L(\frac{4}{5},\frac{1}{15})+L(\frac{4}{5},\frac{2}{3})$ and (2.3) $\sim$ (2.6), we have

$$W(\frac{2}{5})\times W(\frac{1}{15},\pm)=W(\frac{1}{15},\pm)+W(\frac{2}{3},\pm). \tag{3.4}$$

By the fusion rules $L(\frac{4}{5},\frac{2}{5})\times L(\frac{4}{5},\frac{2}{5})=L(\frac{4}{5},0)+L(\frac{4}{5},\frac{7}{5})$ and (2.1) and (2.2), we have

$$W(\frac{2}{5})\times W(\frac{2}{5})=W(0)+W(\frac{2}{5}). \tag{3.5}$$

By the fusion rules $L(\frac{4}{5},\frac{2}{3})\times L(\frac{4}{5},\frac{1}{15})=L(\frac{4}{5},\frac{2}{5})+L(\frac{4}{5},\frac{7}{5})+L(\frac{4}{5},\frac{1}{15})$ and (2.3),(2.5) and (2.8), we have

$$\begin{aligned}&W(\tfrac{2}{3},\pm)\times W(\tfrac{1}{15},\pm)=W(\tfrac{1}{15},\mp)\\&W(\tfrac{2}{3},\pm)\times W(\tfrac{1}{15},\mp)=W(\tfrac{2}{5}).\end{aligned} \tag{3.6}$$

We put the above fusion rules in the following table.

Table B

$W(0)$	$W(\frac{2}{5})$	$W(\frac{2}{3},+)$	$W(\frac{1}{15},+)$	$W(\frac{2}{3},-)$	$W(\frac{1}{15}.-)$
$W(\frac{2}{5})$	$W(0):W(\frac{2}{5})$	$W(\frac{1}{15},+)$	$W(\frac{1}{15},+):W(\frac{2}{3},+)$	$W(\frac{1}{15},-)$	$W(\frac{1}{15},-):W(\frac{2}{3},-)$
$W(\frac{2}{3},+)$	$W(\frac{1}{15},+)$	$W(\frac{2}{3},-)$	$W(\frac{1}{15},-)$	$W(0)$	$W(\frac{2}{5})$
$W(\frac{1}{15},+)$	$W(\frac{1}{15},+):W(\frac{2}{3},+)$	$W(\frac{1}{15},-)$	$W(\frac{1}{15},-):W(\frac{2}{3},-)$	$W(\frac{2}{5})$	$W(0):W(\frac{2}{5})$
$W(\frac{2}{3},-)$	$W(\frac{1}{15},-)$	$W(0)$	$W(\frac{2}{5})$	$W(\frac{2}{3},+)$	$W(\frac{1}{15},+)$
$W(\frac{1}{15},-)$	$W(\frac{1}{15},-):W(\frac{2}{3},-)$	$W(\frac{2}{5})$	$W(0):W(\frac{2}{5})$	$W(\frac{1}{15},+)$	$W(\frac{1}{15},+):W(\frac{2}{3},+)$

4 Automorphisms

As we showed in [M2], if a VOA contains $L(\frac{4}{5},0)$, then we have an automorphism σ of at most 2 given by

$$\sigma:\begin{cases}\ \ 1 & on \quad L(\frac{4}{5},0),L(\frac{4}{5},3),L(\frac{4}{5},\frac{2}{3}),L(\frac{4}{5},\frac{2}{5}),L(\frac{4}{5},\frac{1}{15}),L(\frac{4}{5},\frac{7}{5})\\ -1 & on \quad L(\frac{4}{5},\frac{1}{8}),L(\frac{4}{5},\frac{13}{8}),L(\frac{4}{5},\frac{1}{40}),L(\frac{4}{5},\frac{21}{40})\end{cases}.$$

So we next think about the case $\sigma=1$ or the space V^{σ} of σ-invariants. In this case, there are no $L(\frac{4}{5},0)$-submodules isomorphic to $L(\frac{4}{5},\frac{1}{8}),L(\frac{4}{5},\frac{13}{8}),L(\frac{4}{5},\frac{1}{40})$ or $L(\frac{4}{5},\frac{21}{40})$. We next assume that V contains $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$. We should note that if V contains $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$, then there are no $L(\frac{4}{5},0)$-submodules in V isomorphic to $L(\frac{4}{5},\frac{1}{8}),L(\frac{4}{5},\frac{13}{8}),L(\frac{4}{5},\frac{1}{40})$

or $L(\frac{4}{5},\frac{21}{40})$.

Theorem A *If a VOA V contains $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$, then an endomorphism τ of V defined by*

$$\tau : \begin{cases} 1 & \text{on} \quad W(0) \text{ and } W(\frac{2}{5}) \\ e^{2\pi i/3} & \text{on} \quad W(\frac{2}{3},+) \text{ and } W(\frac{1}{15},+) \\ e^{4\pi i/3} & \text{on} \quad W(\frac{2}{3},-) \text{ and } W(\frac{1}{15},-) \end{cases}$$

is an automorphism of V.

[Proof] Replacing $W(i)$ and $W(h,+)$ and $W(k,-)$ in the table (B) by 1 and $e^{2\pi i/3}$ and $e^{4\pi i/3}$, we have

1	1	$e^{2\pi i/3}$	$e^{2\pi i/3}$	$e^{4\pi i/3}$	$e^{4\pi i/3}$
1	$1:1$	$e^{2\pi i/3}$	$e^{2\pi i/3}:e^{2\pi i/3}$	$e^{4\pi i/3}$	$e^{4\pi i/3}:e^{4\pi i/3}$
$e^{2\pi i/3}$	$e^{2\pi i/3}$	$e^{4\pi i/3}$	$e^{4\pi i/3}$	1	1
$e^{2\pi i/3}$	$e^{2\pi i/3}:e^{2\pi i/3}$	$e^{4\pi i/3}$	$e^{4\pi i/3}:e^{4\pi i/3}$	1	$1:1$
$e^{4\pi i/3}$	$e^{4\pi i/3}$	1	1	$e^{2\pi i/3}$	$e^{2\pi i/3}$
$e^{4\pi i/3}$	$e^{4\pi i/3}:e^{4\pi i/3}$	1	$1:1$	$e^{2\pi i/3}$	$e^{2\pi i/3}:e^{2\pi i/3}$

which is compatible with the products. Hence, by Theorem 2.2, τ is an automorphism of V.

Q.E.D.

If $\tau_W=1$, then all T-submodules of V are isomorphic to $L(\frac{4}{5},0)$, $L(\frac{4}{5},3)$, $L(\frac{4}{5},\frac{2}{5})$ or $L(\frac{4}{5},\frac{7}{5})$ for $T\subseteq W$ and $T\cong L(\frac{4}{5},0)$. In this case, we can define another automorphism μ_T of V as follows:

Theorem B *Assume that V contains a sub VOA T isomorphic to $L(\frac{4}{5},0)$ and all T-submodules of V are isomorphic to $L(\frac{4}{5},0)$, $L(\frac{4}{5},3)$, $L(\frac{4}{5},\frac{2}{5})$ or $L(\frac{4}{5},\frac{7}{5})$. Then the endomorphism μ_T defined by*

$$\mu_T : \begin{cases} 1 & \text{on} \quad L(\frac{4}{5},0) \text{ and } L(\frac{4}{5},\frac{7}{5}) \\ -1 & \text{on} \quad L(\frac{4}{5},3) \text{ and } L(\frac{4}{5},\frac{2}{5}) \end{cases}$$

is an automorphism of V.

5 V as a sub VOA-module

The notion of sub VOAs of V in this paper is not the same as in [FZ], where they expected sub VOA W to have the same Virasoro element with V. Our definition of sub VOAs is: $(W = \oplus W_n, Y^W, w_W, \mathbf{1_W})$ is a sub VOA of V if
(1) $(W, Y^W, w_W, \mathbf{1_W})$ is a VOA,
(2) $W \subseteq V$ and $W_n = W \cap V_n$,
(3) $\mathbf{1_W} = \mathbf{1_V}$ and
(4) $Y^W(v,z) = Y(v,z)|_W$ for $v \in W$.

There are several definitions for VOA-modules, but we will include an infinite direct sum of irreducible modules as a VOA-module M. Namely, we don't assume $\dim M_n < \infty$.

Let W be a sub VOA of V and e a Virasoro element of W. Different from the ordinary algebras, it is not obvious that V is a W-module. The problem is whether e_1 acts on V diagonally or not.

The purpose of this section is to show that V is a W-module for W in our cases.

Let V be a VOA and W a sub VOA. Let w and e be Virasoro elements of V and W, respectively.

In particular, $e \in V_2$ and so e_1 keeps the grade of V. By the assumption, $f_1 = w_1 - e_1$ acts on W as 0. Furthermore, for $v \in W$ $Y(e_0v, z)|_W = \frac{d}{dz}Y(v,z)|_W = Y(w_0v,z)|_W$ and so $Y(w_0v - e_0v, z)|_W = 0$. In particular, $w_0v - e_0v = (w_0v - e_0v)_{-1}\mathbf{1} = 0$ and so, $f_0 = w_0 - e_0$ acts on W as 0. Hence we have:

Lemma 5.1 *Both f_1 and f_0 commutes v_m on V for $v \in W$ and $m \in \mathbb{Z}$.*

[Proof] It follows from $[f_0, v_m] = (f_0v)_m = 0$ and $[f_1, v_m] = (f_0v)_{m+1} + (f_1v)_m = 0$.

Q.E.D.

The main purpose in this section is to prove the following theorem:

Theorem 5.1 *If W is rational, then V is a W-module.*

[Proof] Define a module vertex operator $Y^V(v,z)$ of $v \in W$ by the vertex operator of $v \in V$. Clearly, they satisfy the local commutativity and the e_0-derivative property:

$$Y^V(e_0v, z) = Y^V((f_0 + e_0)v, z) = Y(w_0v, z) = \frac{d}{dz}Y(v,z)$$

for $v \in W$. Hence, what we have to do is to prove that V is a direct sum of eigenspaces of e_1. Suppose false. Since f_1 commutes with all v_n for $v \in W$, the eigenspace V_λ and the generalized eigenspace $T_\lambda = \{v \in V | \exists n \in \mathbb{Z} \quad (f_1 - \lambda)^n v = 0\}$ of f_1 with eigenvalue λ is invariant under the actions v_n of $v \in W$. We first prove that the eigenspace V_λ is a direct sum of irreducible W-modules. Since f_1 keeps the grade, it acts on V_n and so we have $V_\lambda = \oplus (V_\lambda)_n$, where $(V_\lambda)_n = V_n \cap V_\lambda$. Since $e_1 = (w_1 - f_1)$ acts on $(V_\lambda)_n$ as $n - \lambda$, V_λ is a W-module. Since W is rational by the assumption, V_λ is a direct sum of irreducible W-modules.

Since f_1 acts on each finite dimensional homogenous spaces V_n, V is a direct sum of generalized eigenspaces of f_1. Hence, there are λ and n such that $T_\lambda \cap V_n \neq V_\lambda \cap V_n$. Take n as a minimal one.

As we explained as above, v_n acts on T_λ / V_λ for $v \in W$ and the eigenspace X_λ of f_1 in T_λ / V_λ is a W-module. By the choice of n, $(X_\lambda)_n \neq 0$. Let $\bar{X}$ be an irreducible W-submodule of X_λ whose lowest degree is n. We should note that since V_λ is an eigenspace of f_1 and $e_1 = w_1 - f_1$, the lowest eigenspace of w_1 in V_λ is the lowest eigenspace of e_1.

Hence there is an irreducible W-submodule $\bar{B}$ of T_λ / V_λ whose lowest degree space $\bar{B}_0$ is in $(V_n + V_\lambda)/V_\lambda$ since W is rational. Let B be its inverse image. Clearly, B contains V_λ and f_1 does not act on B diagonally. Let S is a submodule of V_λ generated by W-submodules which are not isomorphic to $\bar{B}$. Since $(f_1 - \lambda)B \neq 0$ and all submodule of $(f_1 - \lambda)B$ is isomorphic to $\bar{B}$, all composition factors of B/S is isomorphic to $\bar{B}$ as W-modules and we have $S \neq V_\lambda$. In particular, $(B/S) \cap (V_m + S/S) = 0$ for all $m < n$. We next show that Zhu-algebra $A(W)$ acts on the top module $(B_n + S)/S$ of B/S.

In order to prove the above assertion, we will use an idea for Zhu-algebra in [Z]. We will treat a general case for a while. Let $A(V) = V/O(V)$ be the Zhu-algebra of V. For $v \in V$, $o(v)$ denotes the grade keeping operator of v. For a homogeneous element $v \in V_m$, if $Y^M(v, z) = \sum v_i z^{-i-1}$ is a module vertex operator, then $o(v) = v_{m-1}$. It actually depends on the module M, but we write $o(v) = v_{m-1}$ formally.

Let R be the ring generated by all $o(v)$ of $v \in V$. Let I be a subspace of R generated by the elements $\sigma = v^1_{i_1} \cdots v^r_{i_r} \in R$ satisfying that $v^s_{i_s} \cdots v^r_{i_r}$ decreases the grade for some $1 \leq s \leq r$. We permit an infinite sum of such elements if it is well-defined in R. Clearly, I is a two-sided ideal of R. It is known that $A(V) = R/I$, see [Z].

Let's go back to the proof. Since W is rational, $A(W)$ is a semi-simple. Let $\phi = v^1_{i_1} \cdots v^t_{i_t} \in R$ and assume that $v^s_{i_s} \cdots v^t_{i_t}$ decreases the grade on W. Since the grade on W is the same as that on V, ϕ acts on $(B_n + S)/S$ trivially. Hence $A(W)$ acts on $(B_n + S)/S$. Since $A(W)$ is semi-simple, $(B_n + S)/S$ is a direct sum of irreducible $A(W)$-modules. By the definition of B and S, $(B_n + S)/S$ is a homogeneous $A(W)$-module. Since e_1 is in the

center of $A(W)$, e_1 acts on $(B_n+S)/S$ as a scalar times and so does f_1, which contradicts to the facts that $(f_1-\lambda)(B_n+S)/S \neq 0$.

Hence, V is a direct sum of eigenspaces of f_1 and so e_1 acts on V diagonally. This completes the proof of Theorem 5.1.

Q.E.D.

6 Appendix

6.1 Fusion rules of irreducible $L(\frac{4}{5},0)$-modules

For $L(\frac{4}{5},0)$, the following fusion rules are known, see [W]. In the following table, the numbers h denote $L(\frac{4}{5},h)$ and $h_1:...:h_t$ denotes $L(\frac{4}{5},h_1)+...+h(\frac{4}{5},h_t)$.

Table C

0	$\frac{2}{5}$	$\frac{1}{40}$	$\frac{7}{5}$	$\frac{21}{40}$	$\frac{1}{15}$	3	$\frac{13}{8}$	$\frac{2}{3}$	$\frac{1}{8}$
$\frac{2}{5}$	$0:\frac{7}{5}$	$\frac{1}{8}:\frac{21}{40}$	$\frac{2}{5}:3$	$\frac{1}{40}:\frac{13}{8}$	$\frac{1}{15}:\frac{2}{3}$	$\frac{7}{5}$	$\frac{21}{40}$	$\frac{1}{15}$	$\frac{1}{40}$
$\frac{1}{40}$	$\frac{1}{8}:\frac{21}{40}$	$0:\frac{7}{5}:\frac{2}{3}:\frac{1}{15}$	$\frac{1}{40}:\frac{13}{8}$	$\frac{2}{5}:3:\frac{1}{15}:\frac{2}{3}$	$\frac{1}{40}:\frac{13}{8}:\frac{21}{40}:\frac{1}{8}$	$\frac{21}{40}$	$\frac{7}{5}:\frac{1}{15}$	$\frac{21}{40}:\frac{1}{40}$	$\frac{1}{15}:\frac{2}{5}$
$\frac{7}{5}$	$\frac{2}{5}:3$	$\frac{1}{40}:\frac{13}{8}$	$0:\frac{7}{5}$	$\frac{1}{8}:\frac{21}{40}$	$\frac{2}{3}:\frac{1}{15}$	$\frac{2}{5}$	$\frac{1}{40}$	$\frac{1}{15}$	$\frac{21}{40}$
$\frac{21}{40}$	$\frac{1}{40}:\frac{13}{8}$	$\frac{2}{5}:3:\frac{1}{15}:\frac{2}{3}$	$\frac{1}{8}:\frac{21}{40}$	$0:\frac{7}{5}:\frac{2}{3}:\frac{1}{15}$	$\frac{1}{8}:\frac{21}{40}:\frac{13}{8}:\frac{1}{40}$	$\frac{1}{40}$	$\frac{2}{5}:\frac{1}{15}$	$\frac{1}{40}:\frac{21}{40}$	$\frac{1}{15}:\frac{7}{5}$
$\frac{1}{15}$	$\frac{1}{15}:\frac{2}{3}$	$\frac{1}{40}:\frac{13}{8}:\frac{21}{40}:\frac{1}{8}$	$\frac{2}{3}:\frac{1}{15}$	$\frac{1}{8}:\frac{21}{40}:\frac{13}{8}:\frac{1}{40}$	$0:\frac{7}{5}:\frac{2}{3}:\frac{1}{15}:3:\frac{2}{5}$	$\frac{1}{15}$	$\frac{1}{40}:\frac{21}{40}$	$\frac{2}{5}:\frac{1}{15}:\frac{7}{5}$	$\frac{1}{40}:\frac{21}{40}$
3	$\frac{7}{5}$	$\frac{21}{40}$	$\frac{2}{5}$	$\frac{1}{40}$	$\frac{1}{15}$	0	$\frac{1}{8}$	$\frac{2}{3}$	$\frac{13}{8}$
$\frac{13}{8}$	$\frac{21}{40}$	$\frac{7}{5}:\frac{1}{15}$	$\frac{1}{40}$	$\frac{2}{5}:\frac{1}{15}$	$\frac{1}{40}:\frac{21}{40}$	$\frac{1}{8}$	$0:\frac{2}{3}$	$\frac{1}{8}:\frac{13}{8}$	$\frac{2}{3}:3$
$\frac{2}{3}$	$\frac{1}{15}$	$\frac{21}{40}:\frac{1}{40}$	$\frac{1}{15}$	$\frac{1}{40}:\frac{21}{40}$	$\frac{2}{5}:\frac{1}{15}:\frac{7}{5}$	$\frac{2}{3}$	$\frac{1}{8}:\frac{13}{8}$	$0:\frac{2}{3}:3$	$\frac{1}{8}:\frac{13}{8}$
$\frac{1}{8}$	$\frac{1}{40}$	$\frac{1}{15}:\frac{2}{5}$	$\frac{21}{40}$	$\frac{1}{15}:\frac{7}{5}$	$\frac{1}{40}:\frac{21}{40}$	$\frac{13}{8}$	$\frac{2}{3}:3$	$\frac{1}{8}:\frac{13}{8}$	$0:\frac{2}{3}$

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